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Homotopy groups of moduli spaces of representations

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Abstract

We calculate certain homotopy groups of the moduli spaces for representations of a compact oriented surface in the Lie groups $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{U}(p, q)$. Our approach relies on the interpretation of these representations in terms of Higgs bundles and uses Bott–Morse theory on the corresponding moduli spaces.

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1. Introduction

Given a closed oriented surface, X, and a Lie group G, moduli spaces of surface group representations in G have a rich geometric and topological structure which reflects properties of both X and G. In this paper, we consider the cases where G is $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{U}(p, q)$.

Our main tools rely on an interpretation of the moduli spaces in terms of holomorphic bundles. Such an interpretation starts from the basic correspondence between representations of the fundamental group and flat principal bundles. Holomorphic bundles enter the picture if we fix a complex structure on the surface X — thereby turning it into a Riemann surface. By results of Hitchin [18],

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Donaldson [12], Simpson [24] and Corlette [9], if G is complex semisimple then the flat principal G-bundles corresponding to semisimple representations of $\pi_1 X$ in G are equivalent to polystable G-Higgs bundles over the Riemann surface. More generally, such Higgs bundles exist if G is complex reductive, in which case the polystable G-Higgs bundles correspond to semisimple representations not of $\pi_1 X$, but of a central extension of the fundamental group.

Referring to $\pi_1 X$ and its central extensions as surface groups, we can thus identify the moduli spaces of surface group representations with moduli spaces of polystable Higgs bundles. This identification puts a natural Kähler structure on the moduli spaces and also reveals a compatible $\mathbb{C}^*$-action. The restriction of this action to S^1 leads to a symplectic moment map whose squared norm serves as a proper Morse function. In a striking example of the interplay between geometry and topology, these geometric features on the moduli spaces of Higgs bundles provide powerful tools for studying the topology of the underlying moduli spaces of surface group representations.

Holomorphic bundle techniques can also be adapted to the case in which G is a real reductive Lie group, in particular when G is a real form of a complex reductive group. If G is the compact real form of a complex reductive group $G_{\mathbb{C}}$, then the theorem of Narasimhan and Seshadri [22] and its generalization by Ramanathan [23] identify representations into G with polystable principal $G_{\mathbb{C}}$-bundles. For non-compact real forms, the basic ideas were first introduced by Nigel Hitchin. In [19], he outlined how to define the appropriate Higgs bundles and applied his methods to the case $G = \mathrm{SL}(n, \mathbb{R})$ and also to other split real forms. Other special cases have been considered in a similar way.[2] In [5],we began an in-depth study of the groups $\mathrm{U}(p, q)$ and their adjoint forms $\mathrm{PU}(p, q)$ (for any p and q) from this point of view. This paper is a continuation of that work.

The most primitive topological feature of these moduli spaces is their number of connected components, i.e. π_0. The above methods have been effective in addressing this question, mainly by exploiting the properness of the above mentioned Morse function. This transfers questions about π_0 for the moduli spaces into questions about the connected components of the minimal submanifolds for the Morse function.

In good cases, there is additional useful Morse theoretic information which has thus far gone unexploited. Our goal is to correct this oversight. In particular, using information about the Morse indices of non-minimal critical points, we can relate higher homotopy groups for the full moduli spaces to those of their minimal submanifolds. For the latter, we rely on the calculations by Daskalopoulos and Uhlenbeck [10] for higher homotopy groups of the moduli space of stable vector bundles. Our main results for the moduli spaces of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{U}(p, q)$ Higgs bundles, and hence for the corresponding moduli spaces of representations, are given in Theorems 4.4 and 4.22 respectively.

2. Surface group representations and Higgs bundles

For a more thorough account of the material in this section, see [5].

2.1. Surface group representations

Let X be a smooth closed oriented surface of genus $g \geqslant 2$. The fundamental group, $\pi_1 X$, of X is a finitely generated group generated by $2g$ generators, say $A_1, B_1, \ldots, A_g, B_g$, subject to the single

[2] Notably $\mathrm{Sp}(4, \mathbb{R})$ and $\mathrm{SU}(2, 2)$ [15,14], $\mathrm{U}(2, 1)$ [16], $\mathrm{U}(p, q)$ and $\mathrm{PU}(p, q)$ [5], $\mathrm{GL}(n, \mathbb{R})$ [7]. Higgs bundle methods have also been applied, albeit in a more algebraic way, in the cases $\mathrm{U}(p, 1)$ [27], $\mathrm{PU}(2, 1)$ [28], and $\mathrm{PU}(p, p)$ [21].

relation $\prod_{i=1}^{g}[A_i, B_i] = 1$. It has a universal central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1 X \longrightarrow 1 \tag{2.1}$$

generated by the same generators as $\pi_1 X$, together with a central element J subject to the relation $\prod_{i=1}^{g}[A_i, B_i] = J$.

By a representation of Γ in $\mathrm{GL}(n, \mathbb{C})$ we mean a homomorphism $\rho\colon \Gamma \to \mathrm{GL}(n, \mathbb{C})$. We say that a representation of Γ in $\mathrm{GL}(n, \mathbb{C})$ is *semisimple* if the $\mathbb{C}^n$-representation of Γ induced by the fundamental representation of $\mathrm{GL}(n, \mathbb{C})$ is semisimple.[3] The group $\mathrm{GL}(n, \mathbb{C})$ acts on the set of representations via conjugation. Restricting to the semisimple representations, denoted by $\mathrm{Hom}^+(\Gamma, \mathrm{GL}(n, \mathbb{C}))$, we get the *moduli space* of representations of Γ in $\mathrm{GL}(n, \mathbb{C})$,

$$\mathcal{R}(\Gamma, \mathrm{GL}(n, \mathbb{C})) = \mathrm{Hom}^+(\Gamma, \mathrm{GL}(n, \mathbb{C}))/\mathrm{GL}(n, \mathbb{C}). \tag{2.2}$$

The set $\mathrm{Hom}^+(\Gamma, \mathrm{GL}(n, \mathbb{C}))$ can be embedded in $\mathrm{GL}(n, \mathbb{C})^{2g+1}$ via the map

$$\begin{aligned} &\mathrm{Hom}^+(\Gamma, \mathrm{GL}(n, \mathbb{C})) \to \mathrm{GL}(n, \mathbb{C})^{2g+1} \\ &\rho \mapsto (\rho(A_1), \ldots \rho(B_g), \rho(J)). \end{aligned}$$

We can then give $\mathrm{Hom}^+(\Gamma, \mathrm{GL}(n, \mathbb{C}))$ the subspace topology, and $\mathcal{R}(\Gamma, \mathrm{GL}(n, \mathbb{C}))$ the quotient topology. This topology is Hausdorff because we have restricted attention to semisimple representations.

There is a topological invariant of a representation $\rho \in \mathcal{R}(\Gamma, \mathrm{GL}(n, \mathbb{C}))$ given by $\rho(J)$, which coincides with the first Chern class of the vector bundle with central curvature associated to ρ. Fixing this invariant, we define

$$\mathcal{R}(n, d) := \{\rho \in \mathcal{R}(\Gamma, \mathrm{GL}(n, \mathbb{C})) \mid \rho(J) = [d] \in \mathbb{Z}_n \subset Z(\mathrm{GL}(n, \mathbb{C}))\}.$$

In particular, the representations with vanishing degree correspond to representations of the fundamental group of X, that is,

$$\mathcal{R}(n, 0) = \mathcal{R}(\pi_1 X, \mathrm{GL}(n, \mathbb{C})) := \mathrm{Hom}^+(\pi_1 X, \mathrm{GL}(n, \mathbb{C}))/\mathrm{GL}(n, \mathbb{C}). \tag{2.3}$$

Similarly to the case of $\mathrm{GL}(n, \mathbb{C})$ we consider the moduli space

$$\mathcal{R}(\Gamma, \mathrm{U}(p, q)) = \mathrm{Hom}^+(\Gamma, \mathrm{U}(p, q))/\mathrm{U}(p, q). \tag{2.4}$$

The moduli space $\mathcal{R}(\Gamma, \mathrm{U}(p, q))$ can be identified with the moduli space of $\mathrm{U}(p, q)$-bundles on X with projectively flat connections. Taking a reduction to the maximal compact $\mathrm{U}(p) \times \mathrm{U}(q)$, we thus associate to each class $\rho \in \mathcal{R}(\Gamma, \mathrm{U}(p, q))$ a vector bundle of the form $V \oplus W$, where V and W are rank p and q respectively, and thus a pair of integers $(a, b) = (\deg(V), \deg(W))$. There is thus a map

$$c\colon \mathcal{R}(\Gamma, \mathrm{U}(p, q)) \to \mathbb{Z} \oplus \mathbb{Z}$$

given by $c(\rho) = (a, b)$. The corresponding map on $\mathrm{Hom}^+(\Gamma, \mathrm{U}(p, q))$ is clearly continuous, and thus locally constant. Since $\mathrm{U}(p, q)$ is connected, the map c is likewise continuous, and thus constant on connected components. The subspace of $\mathcal{R}(\Gamma, \mathrm{U}(p, q))$ corresponding to representations with invariants

[3] In general, a representation of Γ in a reductive Lie group G is said to be semisimple if the induced (adjoint) representation on the Lie algebra of G is semisimple. For $G = \mathrm{GL}(n, \mathbb{C})$ this is equivalent to the definition given here.

(a, b) is denoted by

$$\mathcal{R}(p, q, a, b) = c^{-1}(a, b) = \{\rho \in \mathcal{R}(\Gamma, \mathrm{U}(p, q)) \mid c(\rho) = (a, b) \in \mathbb{Z} \oplus \mathbb{Z}\}. \tag{2.5}$$

The representations for which $a + b = 0$ correspond to representations of the fundamental group of X, that is,

$$\mathcal{R}(p, q, a, -a) = c^{-1}(a, -a) = \{\rho \in \mathcal{R}(\pi_1 X, \mathrm{U}(p, q)) \mid c(\rho) = (a, -a) \in \mathbb{Z} \oplus \mathbb{Z}\}. \tag{2.6}$$

2.2. $\mathrm{GL}(n, \mathbb{C})$-*Higgs bundles*

A $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle on a compact Riemann surface X is a pair (E, Φ), where E is a rank n holomorphic vector bundle over X and $\Phi \in H^0(\mathrm{End}(E) \otimes K)$ is a holomorphic endomorphism of E twisted by the canonical bundle K of X. The $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle (E, Φ) is *stable* if the slope stability condition

$$\mu(E') < \mu(E) \tag{2.7}$$

holds for all proper Φ-invariant subbundles E' of E. Here, the *slope* is defined by $\mu(E) = \deg(E)/\mathrm{rk}(E)$ and Φ-*invariance* means that $\Phi(E') \subset E' \otimes K$. *Semistability* is defined by replacing the above strict inequality with a weak inequality. A Higgs bundle is called *polystable* if it is the direct sum of stable Higgs bundles with the same slope.

Given a HermitianHermitian metric on E, let A denote the unique unitary connection compatible with the holomorphic structure, and let F_A be its curvature. *Hitchin's equations* on (E, Φ) are

$$\begin{aligned} &F_A + [\Phi, \Phi^*] = -\sqrt{-1}\mu \mathrm{Id}_E \omega, \\ &\bar{\partial}_A \Phi = 0, \end{aligned} \tag{2.8}$$

where ω is the Kähler form on X, Id_E is the identity on E, $\mu = \mu(E)$ and $\bar{\partial}_A$ is the anti-holomorphic part of the covariant derivative d_A. A solution to Hitchin's equations is *irreducible* if there is no proper subbundle of E preserved by A and Φ.

If we define a Higgs connection (as in [25]) by

$$D = d_A + \theta \tag{2.9}$$

where $\theta = \Phi + \Phi^*$, then Hitchin's equations are equivalent to the conditions

$$\begin{aligned} &F_D = -\sqrt{-1}\mu \mathrm{Id}_E \omega, \\ &d_A^* \theta = 0. \end{aligned} \tag{2.10}$$

In particular, the first equation says that D is a projectively flat connection.[4] If $\deg(E) = 0$ then D is actually flat. It follows that in this case, the pair (E, D) defines a representation of $\pi_1 X$ in $\mathrm{GL}(n, \mathbb{C})$. If $\deg(E) \neq 0$, then the pair (E, D) defines a representation of $\pi_1 X$ in $\mathrm{PGL}(n, \mathbb{C})$, or equivalently, a representation of Γ in $\mathrm{GL}(n, \mathbb{C})$. By the theorem of Corlette [9], every semisimple representation of Γ (and therefore every semisimple representation of $\pi_1 X$) arises in this way.

[4] The other equation is a harmonicity constraint.

If we fix the rank and degree (say n and d respectively) of the bundle E, i.e. on bundles of fixed topological type, the isomorphism classes of polystable Higgs bundles are parameterized by a quasi-projective variety of dimension $2+2n^2(g-1)$. We denote this moduli space of rank n degree d polystable Higgs bundles by $\mathcal{M}(n, d)$.

If we fix a Hermitian metric on a smooth rank n degree d complex vector bundle on X, then there is a gauge theoretic moduli space of pairs (A, Φ), consisting of a unitary connection A and an endomorphism valued $(1, 0)$-form Φ, which are solutions to Hitchin's equations (2.8), modulo $\mathrm{U}(n)$-gauge equivalence.

The gauge theory moduli space and $\mathcal{M}(n, d)$ are related by virtue of the Hitchin-Kobayashi correspondence: a $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle (E, Φ) is polystable if and only if it admits a Hermitian metric such that Hitchin's equations (2.8) are satisfied, and (E, Φ) is stable if and only if the corresponding solution is irreducible. There is, moreover, a map from the gauge theoretic moduli space to this moduli space given by taking a solution (A, Φ) to Hitchin's equations to the Higgs bundle (E, Φ), where the holomorphic structure on E is given by $\bar{\partial}_A$. This map is a homeomorphism, and a diffeomorphism on the smooth locus.

In view of the relation between Hitchin's equations and projectively flat connections, this correspondence gives rise to a homeomorphism between $\mathcal{M}(n, d)$ and the component $\mathcal{R}(n, d)$ of the moduli space of semisimple representations of Γ in $\mathrm{GL}(n, \mathbb{C})$. If the degree of the Higgs bundle is zero, then the moduli space $\mathcal{M}(n, 0)$ is homeomorphic to the moduli space of representations of $\pi_1 X$ in $\mathrm{GL}(n, \mathbb{C})$.

Theorem 2.1. *If* (n, d) *is such that* $\mathrm{GCD}(n, d) = 1$*, then the moduli space* $\mathcal{M}(n, d)$ *is a non-empty connected smooth hyperkähler manifold.*

2.3. $\mathrm{U}(p, q)$-*Higgs bundles*

There is a special class of $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles, related to representations in $\mathrm{U}(p, q)$ given by the requirements that

$$\begin{aligned} E &= V \oplus W \\ \Phi &= \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \end{aligned} \tag{2.11}$$

where V and W are holomorphic vector bundles of rank p and q respectively and the non-zero components in the Higgs field are $\beta \in H^0(\mathrm{Hom}(W, V) \otimes K)$, and $\gamma \in H^0(\mathrm{Hom}(V, W) \otimes K)$. We say (E, Φ) is a *stable* $\mathrm{U}(p, q)$-Higgs bundle if the slope stability condition $\mu(E') < \mu(E)$, is satisfied for all Φ-invariant subbundles $E' = V' \oplus W'$, i.e. for all subbundles $V' \subset V$ and $W' \subset W$ such that

$$\beta : W' \longrightarrow V' \otimes K \tag{2.12}$$

$$\gamma : V' \longrightarrow W' \otimes K. \tag{2.13}$$

Semistability and polystability are defined analogously to the way they are defined for $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles.

Hitchin's equations make sense for $\mathrm{U}(p, q)$-Higgs bundles, with a $\mathrm{U}(p, q)$ solution being a metric with respect to which $E = V \oplus W$ is an orthogonal decomposition. With Φ as in (2.11) and $\theta = \Phi + \Phi^*$, the corresponding $\mathrm{U}(p, q)$-Higgs connection $D = d_A + \theta$ is not only projectively flat but has $\mathrm{U}(p, q)$

holonomy. This provides the link between $\mathrm{U}(p,q)$-Higgs bundles and surface group representations in $\mathrm{U}(p,q)$, leading to:

Theorem 2.2. *Let $\mathcal{M}(p,q,a,b)$ be the moduli space of polystable $\mathrm{U}(p,q)$-Higgs bundles with $\deg(V)=a$ and $\deg W=b$. Then with $\mathcal{R}(p,q,a,b)$ as in (2.5) there is a homeomorphism $\mathcal{M}(p,q,a,b)\cong\mathcal{R}(p,q,a,b)$.*

The *Toledo invariant* of the representation corresponding to $(E=V\oplus W,\ \Phi)$ is defined by

$$\tau=\tau(p,q,a,b)=2\frac{qa-pb}{p+q} \tag{2.14}$$

where $a=\deg(V)$ and $b=\deg(W)$. This invariant satisfies the following Milnor-Wood-type inequality (proved by Domic and Toledo [11])

$$|\tau(p,q,a,b)|\leqslant\min\{p,q\}(2g-2). \tag{2.15}$$

Theorem 2.3 (*[5]*). *Let (p,q,a,b) such that $\mathrm{GCD}(p+q,a+b)=1$. Then $\mathcal{M}(p,q,a,b)$ (and hence $\mathcal{R}(p,q,a,b)$) is a connected smooth Kähler manifold which is non-empty if and only if $|\tau(p,q,a,b)|\leqslant\min\{p,q\}(2g-2)$.*

3. Morse theory on the moduli space

3.1. The Morse function

Let $\mathcal{M}$ be either $\mathcal{M}(n,d)$ or $\mathcal{M}(p,q,a,b)$. We will assume that $\mathrm{GCD}(n,d)=1$ and $\mathrm{GCD}(p+q,a+b)=1$. Under this coprimality condition, there are no strictly semistable Higgs bundles and the moduli space $\mathcal{M}$ is smooth. The non-zero complex numbers $\mathbb{C}^*$ act on $\mathcal{M}$ via the map $\lambda\cdot(E,\Phi)=(E,\lambda\Phi)$. However, to have an action on the gauge theory moduli space (i.e. on the set of solutions to Hitchin's equations (2.8), cf. Section 2), one must restrict to the action of $S^1\subset\mathbb{C}^*$. This is a Hamiltonian action and the associated moment map is

$$[(A,\Phi)]\mapsto-\frac{1}{2}\|\Phi\|^2=-i\int_X\mathrm{tr}(\Phi\Phi^*)$$

where the adjoint Φ^* is taken with respect to the Hermitian metric on E. We shall, however, prefer to consider the positive function

$$f([A,\Phi])=\frac{1}{2}\|\Phi\|^2. \tag{3.1}$$

Next we recall a general result of Frankel [13], which was first used in the context of moduli spaces of Higgs bundles by Hitchin [18].

Theorem 3.1. *Let $\tilde f\colon M\to\mathbb{R}$ be a proper moment map for a Hamiltonian circle action on a Kähler manifold M. Then $\tilde f$ is a perfect Bott–Morse function.*

3.2. Morse theory and homotopy groups

In this section, we recall some basic facts of the Bott–Morse theory. Let $\mathcal{M}_l \subset \mathcal{M}$ be the critical submanifolds of f and $\nu(\mathcal{M}_l)$ be the normal bundle of $\mathcal{M}_l$ in $\mathcal{M}$. The Hessian of f is non-degenerate on $\nu(\mathcal{M}_l)$ and we have the decomposition in positive and negative eigenspace bundles

$$\nu(\mathcal{M}_l) = \nu^+(\mathcal{M}_l) \oplus \nu^-(\mathcal{M}_l).$$

The index of $\mathcal{M}_l$ is defined as

$$\mathrm{index}(\mathcal{M}_l) := \mathrm{rk}\nu^-(\mathcal{M}_l).$$

Let $\mathcal{M}_l^+$ be the stable set of $\mathcal{M}_l$, i.e., the subset of $\mathcal{M}$ defined by the points of $\mathcal{M}$ which flow to $\mathcal{M}_l$. It follows from the Bott–Morse theory that $\mathcal{M}_l^+$ is a submanifold of $\mathcal{M}$ of codimension

$$\mathrm{codim}_{\mathbb{R}}(\mathcal{M}_l^+) = \mathrm{index}(\mathcal{M}_l), \tag{3.2}$$

and that there is a stratification

$$\mathcal{M} = \bigcup_l \mathcal{M}_l^+. \tag{3.3}$$

Proposition 3.2. *Let $\mathcal{N} = \mathcal{M}_0 \subset \mathcal{M}$ be the submanifold of local minima of f. If* $\mathrm{index}(\mathcal{M}_l) \geqslant m \geqslant 2$ *for every $l \neq 0$, then*

$$\pi_i(\mathcal{M}) \cong \pi_i(\mathcal{N}) \quad \text{for } i \leqslant m-2.$$

Proof. The stratification (3.3) shows that

$$\mathcal{M}_0^+ = \mathcal{M} \setminus \bigcup_{l\neq 0} \mathcal{M}_l^+$$

and the Morse flow defines a retraction from $\mathcal{M}_0^+$ to $\mathcal{N} = \mathcal{M}_0$. Thus, the result is an immediate consequence of standard homotopy theory, using (3.2). □

3.3. Deformation theory of Higgs bundles

In the following, we recall some standard facts about the deformation theory of Higgs bundles (this has been treated in many places, a convenient reference is Biswas–Ramanan [2]). In order to describe the results in a uniform way for a G-Higgs bundle (E, Φ) when $G = \mathrm{GL}(n,\mathbb{C})$ or $\mathrm{U}(p,q)$, we introduce bundles U_G^+, U_G^- and U_G defined by

$$\begin{aligned}
&U^+_{\mathrm{GL}(n,\mathbb{C})} = U^-_{\mathrm{GL}(n,\mathbb{C})} = U_{\mathrm{GL}(n,\mathbb{C})} = \mathrm{End}(E),\\
&U^+_{\mathrm{U}(p,q)} = \mathrm{End}(V) \oplus \mathrm{End}(W),\\
&U^-_{\mathrm{U}(p,q)} = \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W),\\
&U_{\mathrm{U}(p,q)} = U^+_{\mathrm{U}(p,q)} \oplus U^-_{\mathrm{U}(p,q)} = \mathrm{End}(V \oplus W),
\end{aligned}$$

where the bundles V and W are as in Section 2.3. Note that, with this notation, $\Phi \in H^0(U_G^- \otimes K)$.

Remark 3.3. Both for $G = \mathrm{GL}(n, \mathbb{C})$ and for $G = \mathrm{U}(p, q)$, there is an inner product on U_G which is invariant under the adjoint action of U_G, i.e.,

$$\langle \mathrm{ad}(\psi)x, y\rangle + \langle x, \mathrm{ad}(\psi)y\rangle = 0 \tag{3.4}$$

for local sections x, y and ψ of U_G. This inner product restricts to an inner product on U_G^- and U_G^+, giving rise to an isomorphism

$$U_G^\pm \xrightarrow{\cong} (U_G^\pm)^*. \tag{3.5}$$

Note that under this duality

$$\mathrm{ad}(\Phi)^t = -\mathrm{ad}(\Phi) \otimes 1_{K^{-1}}.$$

Proposition 3.4. *Let (E, Φ) be a G-Higgs bundle for $G = \mathrm{GL}(n, \mathbb{C})$ or $G = \mathrm{U}(p, q)$, and define the following complex of sheaves*

$$C_G^\bullet(E, \Phi)\colon U_G^+ \xrightarrow{\mathrm{ad}(\Phi)} U_G^- \otimes K.$$

Then the following holds:

(1) *The space of endomorphisms of (E, Φ) is naturally isomorphic to $\mathbb{H}^0(C_G^\bullet)$.*
(2) *The infinitesimal deformation space of (E, Φ) is naturally isomorphic to $\mathbb{H}^1(C_G^\bullet)$.*

The following proposition is simply a statement of the fact that a stable Higgs bundle is simple.

Proposition 3.5. *Let (E, Φ) be a stable G-Higgs bundle for $G = \mathrm{GL}(n, \mathbb{C})$ or $G = \mathrm{U}(p, q)$. Then*

$$\mathbb{H}^0(C_G^\bullet(E, \Phi)) \cong \mathbb{C},$$

is generated by the scalar multiples of the identity morphism.

3.4. Critical points and Morse indices

In the following, (E, Φ) continues to denote a G-Higgs bundle for $G = \mathrm{GL}(n, \mathbb{C})$ or $G = \mathrm{U}(p, q)$ and for ease of notation we omit the subscript G on the bundles $U_G^\pm$ and the complex $C_G^\bullet$. The critical points of the function f are exactly the fixed points of the S^1-action on $\mathcal{M}$. This allows one to describe the corresponding Higgs bundles as "complex variations of Hodge structure", as follows (cf. Hitchin [18, 19] and also Simpson [25]).

Proposition 3.6. *If (E, Φ) corresponds to a critical point of f, then there is a semisimple element $\psi \in H^0(U^+)$ and a corresponding decomposition in eigenspace bundles*

$$U_G^\pm = \bigoplus_k U_k^\pm \tag{3.6}$$

for the adjoint action of ψ, such that $\mathrm{ad}(\psi)$ has eigenvalue ik on $U_k^\pm$. Furthermore, $[\psi, \Phi] = i\Phi$, i.e.,

$$\Phi \in H^0(U_1^- \otimes K).$$

In particular, this means that the deformation complex of (E, Φ) decomposes as

$$C^\bullet(E, \Phi) = \bigoplus_k C_k^\bullet(E, \Phi), \tag{3.7}$$

where we have defined for each k the complex

$$C_k^\bullet(E, \Phi)\colon U_k^+ \xrightarrow{\mathrm{ad}(\Phi)} U_{k+1}^- \otimes K.$$

Thus, the tangent space to $\mathcal{M}$ at (E, Φ) has a decomposition

$$\mathbb{H}^1(C^\bullet(E, \Phi)) = \bigoplus_k \mathbb{H}^1(C_k^\bullet(E, \Phi)). \tag{3.8}$$

Remark 3.7. Using the definition of the U_k and (3.4), we have that

$$U_k^\pm \cong U_{-k}^{\pm,*}$$

under the duality (3.5). Moreover, writing

$$\mathrm{ad}(\Phi)_k^\pm = \mathrm{ad}(\Phi)_{|U_k^\pm}\colon U_k^\pm \to U_{k+1}^\mp \otimes K,$$

we have

$$\mathrm{ad}(\Phi)_{k,t}^\pm = (\mathrm{ad}(\Phi)_{-k-1}^\mp) \otimes 1_{K^{-1}}.$$

The calculations of Hitchin [19, Section 8] show that eigenvalues of the Hessian of f at a critical point can be calculated as follows.

Proposition 3.8. *Let (E, Φ) be a stable G-Higgs bundle which corresponds to a critical point of f, for $G = \mathrm{GL}(n, \mathbb{C})$ or $G = \mathrm{U}(p, q)$. In the decomposition (3.8), the eigenvalue $-k$ subspace for the Hessian of f is isomorphic to $\mathbb{H}^1(C_k^\bullet(E, \Phi))$. In particular, the negative eigenspace at (E, Φ) for the Hessian is given by*

$$\nu_{(E,\Phi)}^-(\mathcal{M}_l) \cong \bigoplus_{k>0} \mathbb{H}^1(C_k^\bullet).$$

Lemma 3.9. *Let (E, Φ) be a stable G-Higgs bundle which corresponds to a critical point of f. Then*

$$\mathbb{H}^0(C_k^\bullet(E, \Phi)) = 0 \qquad \text{and} \qquad \mathbb{H}^2(C_k^\bullet(E, \Phi)) = 0$$

for $k > 0$.

Proof. From (3.7), we have a decomposition

$$\mathbb{H}^0(C^\bullet(E, \Phi)) = \bigoplus_k \mathbb{H}^0(C_k^\bullet(E, \Phi))$$

and we know from Proposition 3.5 that the only trivial endomorphisms of (E, Φ) are the scalars, which have weight zero in this decomposition. This gives the vanishing of $\mathbb{H}^0$.

For the vanishing of $\mathbb{H}^2$, consider first the case $G = \mathrm{GL}(n, \mathbb{C})$. Then $U_k^+ = U_k^-$ and, using Remark 3.7, we see that the dual complex of $C_k^\bullet(E, \Phi)$ is isomorphic to the complex

$$C_{-k-1}^\bullet(E, \Phi) \otimes K^{-1} \colon U_{-k-1}^+ \otimes K^{-1} \xrightarrow{-\mathrm{ad}(\Phi)} U_{-k}^-.$$

The change in sign of $\mathrm{ad}(\Phi)$ does not influence the cohomology and hence the Serre duality for hypercohomology gives

$$\mathbb{H}^2(C_k^\bullet(E, \Phi)) \cong \mathbb{H}^0(C_{-k-1}^\bullet(E, \Phi))^*.$$

It follows that $\mathbb{H}^2(C_k^\bullet(E, \Phi))$ vanishes for $k \neq -1$. The case $G = \mathrm{U}(p, q)$ follows essentially from this, by using the fact that stability as a $\mathrm{U}(p, q)$-Higgs bundle implies stability as a $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle (see [5, Proposition 3.19] for a detailed argument). □

Proposition 3.10. *Let (E, Φ) be a stable G-Higgs bundle which corresponds to a critical point of f. Then the Morse index of the corresponding critical submanifold $\mathcal{M}_l$ is*

$$\mathrm{index}(\mathcal{M}_l) = 2 \sum_{k>0} \dim \mathbb{H}^1(C_k^\bullet(E, \Phi)),$$

where

$$\dim \mathbb{H}^1(C_k^\bullet(E, \Phi)) = -\chi(C_k^\bullet(E, \Phi)).$$

Proof. This is immediate from Proposition 3.8 and the vanishing of Lemma 3.9 (note that the Morse index is the real dimension of the $\sum \mathbb{H}^1$, hence the factor of 2). □

The following lemma is essentially Proposition 4.14 of [5]. We provide a complete proof, taking this opportunity to correct some inaccuracies in the argument given in [5].

Lemma 3.11. *Let (E, Φ) be a stable G-Higgs bundle which corresponds to a critical point of f, for $G = \mathrm{GL}(n, \mathbb{C})$ or $G = \mathrm{U}(p, q)$. Then*

$$\chi(C_k^\bullet(E, \Phi)) \leqslant (g-1)(2\mathrm{rk}(\mathrm{ad}(\Phi)_k^+) - \mathrm{rk}(U_k^+) - \mathrm{rk}(U_{k+1}^-)).$$

Furthermore, the vanishing $\chi(C_k^\bullet(E, \Phi)) = 0$ holds if and only if $\mathrm{ad}(\Phi)_k^+ \colon U_k^+ \to U_{k+1}^- \otimes K$ is an isomorphism.

Proof. In the following, we shall use the abbreviated notations $C_k^\bullet = C_k^\bullet(E, \Phi)$ and

$$\Phi_k^\pm = \mathrm{ad}(\Phi)_k^\pm \colon U_k^+ \to U_{k-1}^- \otimes K.$$

By the Riemann–Roch formula, we have

$$\chi(C_k^\bullet) = (1-g)(\mathrm{rk}(U_k^+) + \mathrm{rk}(U_{k+1}^-)) + \deg(U_k^+) - \deg(U_{k+1}^-). \tag{3.9}$$

Thus we can prove the inequality stated in the lemma by estimating the difference $\deg(U_k^+) - \deg(U_k^-)$. In order to do this, we note first that there are short exact sequences of sheaves

$$0 \to \ker(\Phi_k^+) \to U_k^+ \to \mathrm{im}(\Phi_k^+) \to 0$$

and

$$0 \to \mathrm{im}(\Phi_k^+) \to U_{k+1}^- \otimes K \to \mathrm{coker}(\Phi_k^+) \to 0.$$

It follows that

$$\deg(U_k^+) - \deg(U_{k+1}^-) = \deg(\ker(\Phi_k^+)) + (2g-2)\mathrm{rk}(U_{k+1}^-) - \deg(\mathrm{coker}(\Phi_k^+)). \tag{3.10}$$

We shall prove the following inequalities below.

$$\deg(\ker(\Phi_k^+)) \leqslant 0, \tag{3.11}$$

$$-\deg(\mathrm{coker}(\Phi_k^+)) \leqslant (2g-2)(-\mathrm{rk}(U_{k+1}^-) + \mathrm{rk}(\Phi_k^+)). \tag{3.12}$$

Combining these with (3.10), we obtain

$$\deg(U_k^+) - \deg(U_{k+1}^-) \leqslant (2g-2)\mathrm{rk}(\Phi_k^+),$$

which, together with (3.9), proves the inequality stated in the lemma.

It remains to prove (3.11) and (3.12). For this we use the fact that the adjoint Higgs bundle $(U_G, \mathrm{ad}(\Phi))$ is semistable (one way of seeing this is to note that it supports a solution to Hitchin's equations). Clearly, the subbundle $\ker(\Phi_k^+) \subseteq U_G$ is $\mathrm{ad}(\Phi)$-invariant, and hence

$$\deg(\ker(\Phi_k^+)) \leqslant \deg(U_G) = 0,$$

thus proving (3.11).

In order to prove (3.12), a bit more work needs to be done. Consider the dual of Φ_k^+,

$$\Phi_k^{+,t} : U_{k+1}^{-,*} \otimes K^{-1} \to U_k^{+,*},$$

and note that the image of Φ_k^+ goes to zero under the restriction map

$$U_{k+1}^- \otimes K \to \ker(\Phi_k^{+,t})^*.$$

Hence, there is an induced map

$$\mathrm{coker}(\Phi_k^+) \to \ker(\Phi_k^{+,t})^*$$

which is generically an isomorphism — in fact, its kernel is the torsion subsheaf of $\mathrm{coker}(\Phi_k^+)$. It follows that

$$\deg(\mathrm{coker}(\Phi_k^+)) \geqslant \deg(\ker(\Phi_k^{+,t})^*).$$

Since $\ker(\Phi_k^{+,t})$ is locally free (in fact a subbundle), this shows that

$$-\deg(\mathrm{coker}(\Phi_k^+)) \leqslant \deg(\ker(\Phi_k^{+,t})), \tag{3.13}$$

the difference being the degree of the torsion subsheaf of $\mathrm{coker}(\Phi_k^+)$. Now Remark 3.7 tells us that we have a commutative diagram

$$\begin{array}{ccc}
U_{k+1}^{-,*} \otimes K^{-1} & \xrightarrow{\Phi_k^{+,t}} & U_k^{+,*} \\
\cong\downarrow & & \cong\downarrow \\
U_{-k-1}^{-} \otimes K^{-1} & \xrightarrow{-\Phi_{-k-1}^{-} \otimes 1_{K^{-1}}} & U_k^{+},
\end{array}$$

and thus

$$\ker(\Phi_k^{+,t}) \cong \ker(\Phi_{-k-1}^-) \otimes K^{-1}$$

from which we conclude that

$$\deg(\ker(\Phi_k^{+,t})) = \deg(\ker(\Phi_{-k-1}^{-})) - (2g-2)\mathrm{rk}(\ker(\Phi_{-k-1}^{-})).$$

Again we apply the semistability of $(U_G, \mathrm{ad}(\Phi))$ to the $\mathrm{ad}(\Phi)$-invariant subbundle $\ker(\Phi_{-k-1}^{-})$ to obtain

$$\deg(\ker(\Phi_k^{+,t})) \leqslant -(2g-2)\mathrm{rk}(\ker(\Phi_{-k-1}^{-})). \tag{3.14}$$

But clearly, $\mathrm{rk}(\Phi_k^{+}) = \mathrm{rk}(\Phi_k^{+,t}) = \mathrm{rk}(\Phi_{-k-1}^{-})$ and $\mathrm{rk}(U_{k+1}^{-}) = \mathrm{rk}(U_{-k-1}^{-,*}) = \mathrm{rk}(U_{-k-1}^{-})$ so

$$\mathrm{rk}(\ker(\Phi_{-k-1}^{-})) = \mathrm{rk}(U_{k+1}^{-}) - \mathrm{rk}(\Phi_k^{+}).$$

Combining this fact with (3.13) and (3.14) concludes the proof of (3.12).

Finally, to prove the last statement of the Lemma, we note that if $\chi(C_k^{\bullet}) = 0$ then $\mathrm{rk}(\Phi_k^{+}) = \mathrm{rk}(U_k^{+}) = \mathrm{rk}(U_{k+1}^{-} \otimes K)$ and equality holds in (3.13), thus showing that Φ_k^{+} is an isomorphism. □

Proposition 3.12.

(1) *For* $\mathcal{M} = \mathcal{M}(p,q,a,b)$

$$\mathrm{index}(\mathcal{M}_l) \geqslant 2g-2$$

for every non-minimal critical submanifold $\mathcal{M}_l \subset \mathcal{M}$.

(2) *For* $\mathcal{M} = \mathcal{M}(n,d)$

$$\mathrm{index}(\mathcal{M}_l) \geqslant (n-1)(2g-2)$$

for every non-minimal critical submanifold $\mathcal{M}_l \subset \mathcal{M}$.

Proof. (1) Let k_0 be the largest k such that $\chi(C_k^{\bullet}(E,\Phi)) \neq 0$. Since $\mathcal{M}_l$ is non-minimal, by Proposition 3.10 we have $k_0 > 0$. The proof of [5, Proposition 4.17] shows that the restriction of $\mathrm{ad}(\Phi)_k^{+} \colon U_k^{+} \to U_{k+1}^{-} \otimes K$ to a fibre is never an isomorphism (in the notation of that proof, $k_0 = m-1$). Hence, the right hand side of the inequality of Lemma 3.9 is strictly negative. Now the result follows from this inequality and Proposition 3.10.

(2) We recall (cf. [18,19,25]) that the decomposition $U = \bigoplus U_k$ comes from a decomposition $E = E_1 \oplus \cdots \oplus E_m$ with $U_k = \bigoplus_{k=j-i} \mathrm{Hom}(E_i, E_j)$. In particular, the weights k are consecutive integers. Thus Proposition 3.10, together with Riemann–Roch and the fact that $U_G^{+} = U_G^{-}$ for $G = \mathrm{GL}(n,\mathbb{C})$, gives

$$\begin{aligned}\frac{1}{2}\mathrm{index}(\mathcal{M}_l) &= (g-1)\sum_{k\geqslant 1}(\mathrm{rk}(U_k) + \mathrm{rk}(U_{k+1}) - 2\mathrm{rk}(\mathrm{ad}(\Phi)_k)) \\ &= (g-1)(\mathrm{rk}(U_1) + 2\mathrm{rk}(U_{k\geqslant 2}) - 2\mathrm{rk}(\mathrm{ad}(\Phi)_{k\geqslant 1})).\end{aligned}$$

But clearly the rank of $\mathrm{ad}(\Phi)_{k\geqslant 1} \colon U_{k\geqslant 1} \to U_{k\geqslant 2} \otimes K$ is less than or equal to the rank of $U_{k\geqslant 2}$, and hence

$$\frac{1}{2}\mathrm{index}(\mathcal{M}_l) \geqslant (g-1)\mathrm{rk}(U_1).$$

Let $\nu_i = \mathrm{rk}(E_i)$. Then $\sum \nu_i = n$ and $\mathrm{rk}(U_1) = \nu_1\nu_2 + \cdots + \nu_{m-1}\nu_m$. One easily shows that $\nu_1\nu_2 + \cdots + \nu_{m-1}\nu_m \geqslant n-1$. This finishes the proof of (2). □

Remark 3.13. Our initial estimate in (2) of Proposition 3.12 was index$(\mathcal{M}_l) \geqslant 2g-2$. It was pointed out to us by an anonymous referee that this could be improved, and also that an alternative way of proving this estimate is as follows. The absolute minimum of f on $\mathcal{M}(n,d)$ is $M(n,d)$, so $H^*(M(n,d))$ injects into $H^*(\mathcal{M}(n,d))$ because f is perfect. For the same reason, any critical submanifold of index l gives a non-trivial contribution to the cohomology of $\mathcal{M}(n,d)$ in dimension l, which is not in the image of $H^*(M(n,d))$. Now Markman [20] shows that $H^*(B\bar{\mathcal{G}})$ (the cohomology of the classifying space of the reduced gauge group) surjects onto $H^*(\mathcal{M}(n,d))$. On the other hand, Uhlenbeck–Daskalopoulos [10] prove that $H^r(B\bar{\mathcal{G}})$ is isomorphic to $H^r(M(n,d))$ for $r < (2g-2)(n-1)$. Hence no critical submanifold can have index $l < (2g-2)(n-1)$.

3.5. Local minima

The minima of the Morse function on $\mathcal{M}(n,d)$ are given by the following [18].

Proposition 3.14. *Let $\mathcal{N}(n,d) \subset \mathcal{M}(n,d)$ be the set of local minima. Then*

$$\mathcal{N}(n,d) = \{(E,\Phi) \in \mathcal{M}(n,d) \mid \Phi = 0\}.$$

Hence $\mathcal{N}(n,d)$ coincides with $M(n,d)$, the moduli space of semistable vector bundles of rank n and degree d, which equals the subvariety $M^s(n,d) \subset M(n,d)$ corresponding to stable bundles if $\mathrm{GCD}(n,d) = 1$.

The minima of the Morse function on $\mathcal{M}(p,q,a,b)$ have been characterized in [5]. One has the following.

Proposition 3.15. *Let $\mathcal{N}(p,q,a,b) \subset \mathcal{M}(p,q,a,b)$ be the set of local minima. Then*

$$\mathcal{N}(p,q,a,b) = \{(E,\Phi) \in \mathcal{M}(p,q,a,b) \mid \beta = 0 \text{ or } \gamma = 0\}.$$

More precisely, let $(E,\Phi) \in \mathcal{N}(p,q,a,b)$. Then

(1) $\beta = 0$ *if and only if $a/p > b/q$ (i.e. $\tau > 0$).*
(2) $\gamma = 0$ *if and only if $a/p < b/q$ (i.e. $\tau < 0$).*

Remark 3.16. Since we are assuming $\mathrm{GCD}(p+q, a+b) = 1$, we have $\tau \neq 0$.

4. Homotopy groups

4.1. Homotopy groups of $\mathcal{M}(n,d)$

Combining Propositions 3.2, 3.12 and 3.14 we have the following.

Theorem 4.1. *Let* $\mathrm{GCD}(n,d) = 1$. *Then*

$$\pi_i(\mathcal{M}(n,d)) \cong \pi_i(M(n,d)), \quad \text{for } i \leqslant 2(g-1)(n-1)-2.$$

Now, the homotopy groups of $M(n,d)$ have been computed by Daskalopoulos and Uhlenbeck [10] (here n and d are not assumed to be coprime). Their result is the following.

Theorem 4.2. *Let $M^s(n,d)$ be the moduli space of stable vector bundles of rank n and degree d. Assume that $n > 1$ and $(n,g) \neq (2,2)$. Then*

(1) $\pi_1(M^s(n,d)) \cong H_1(X,\mathbb{Z})$;
(2) $\pi_2(M^s(n,d)) \cong \mathbb{Z} \oplus \mathbb{Z}_{\mathrm{GCD}}(n,d)$;
(3) $\pi_i(M^s(n,d)) \cong \pi_{i-1}(\mathcal{G})$, *for* $2 < i \leqslant 2(g-1)(n-1)-2$, *where $\mathcal{G}$ is the unitary gauge group.*

Remark 4.3. The proof of (1) when n and d are coprime is given by Atiyah–Bott [1].

As a corollary of Theorems 4.1 and 4.2, we have the following.

Theorem 4.4. *Assume that $n > 1$ and* $\mathrm{GCD}(n,d) = 1$ *and let $g \geqslant 3$. Then*

(1) $\pi_1(\mathcal{M}(n,d)) \cong H_1(X,\mathbb{Z})$;
(2) $\pi_2(\mathcal{M}(n,d)) \cong \mathbb{Z}$;
(3) $\pi_i(\mathcal{M}(n,d)) \cong \pi_{i-1}(\mathcal{G})$, *for* $2 < i \leqslant 2(g-1)(n-1)-2$.

Remark 4.5. As a consequence of Theorem 4.1 and the connectedness of $M(n,d)$ [22], one obtains that $\mathcal{M}(n,d)$ is also connected [18,26].

A proof of (1) when $n = 2$ is given by Hitchin [18].

Remark 4.6. When $n = 2$, Hausel [17, Theorem 7.5.7] proved that the isomorphism (3) holds for $i \leqslant 4g-8$, which is twice as good as our estimate. It would be very interesting to see if this result can be generalized to higher n.

4.2. Moduli space of triples

The next step is to identify the spaces $\mathcal{N}(p,q,a,b)$ as moduli spaces in their own right. They turn out to be examples of the moduli spaces of triples studied in [4,5] and [6]. We briefly recall the relevant definitions and results. See [5] for details.

A *holomorphic triple* on X, $T = (E_1, E_2, \phi)$, consists of two holomorphic vector bundles E_1 and E_2 on X and a holomorphic map $\phi\colon E_2 \to E_1$. Denoting the ranks E_1 and E_2 by n_1 and n_2, and their degrees by d_1 and d_2, we refer to (n_1,n_2,d_1,d_2) as the *type* of the triple.

A homomorphism from $T' = (E'_1, E'_2, \phi')$ to $T = (E_1, E_2, \phi)$ is a commutative diagram

$$\begin{array}{ccc} E'_2 & \xrightarrow{\phi'} & E'_1 \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{\phi} & E_1. \end{array}$$

$T' = (E'_1, E'_2, \phi')$ is a subtriple of $T = (E_1, E_2, \phi)$ if the homomorphisms of sheaves $E'_1 \to E_1$ and $E'_2 \to E_2$ are injective.

For any $\alpha \in \mathbb{R}$, the *α-degree* and *α-slope* of T are defined to be

$$\begin{aligned} \deg_\alpha(T) &= \deg(E_1) + \deg(E_2) + \alpha \mathrm{rk}(E_2), \\ \mu_\alpha(T) &= \frac{\deg_\alpha(T)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)} \\ &= \mu(E_1 \oplus E_2) + \alpha \frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)}. \end{aligned}$$

The triple $T = (E_1, E_2, \phi)$ is α-*stable* if

$$\mu_\alpha(T') < \mu_\alpha(T) \tag{4.1}$$

for any proper sub-triple $T' = (E_1', E_2', \phi')$. Define α-*semistability* by replacing (4.1) with a weak inequality. A triple is called α-*polystable* if it is the direct sum of α-stable triples of the same α-slope. It is *strictly* α-*semistable (polystable)* if it is α-semistable (polystable) but not α-stable.

We denote the moduli space of isomorphism classes of α-*polystable* triples of type (n_1, n_2, d_1, d_2) by

$$\mathcal{N}_\alpha = \mathcal{N}_\alpha(n_1, n_2, d_1, d_2). \tag{4.2}$$

Using Jordan–Hölder filtrations of α-semistable triples, one can define S-equivalence and view $\mathcal{N}_\alpha$ as the moduli space of S-equivalence classes of α-semistable triples. The isomorphism classes of α-*stable* triples form a subspace which we denote by $\mathcal{N}_\alpha^s$.

Proposition 4.7 ([4]). *The moduli space $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is a complex projective variety. A necessary condition for the moduli space $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ to be non-empty is*

$$\begin{cases} 0 \leqslant \alpha_m \leqslant \alpha \leqslant \alpha_M & \text{if } n_1 \neq n_2 \\ 0 \leqslant \alpha_m \leqslant \alpha & \text{if } n_1 = n_2 \end{cases} \tag{4.3}$$

where

$$\alpha_m = \mu_1 - \mu_2, \tag{4.4}$$

$$\alpha_M = \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right)(\mu_1 - \mu_2) \tag{4.5}$$

and $\mu_1 = \frac{d_1}{n_1}$, $\mu_2 = \frac{d_2}{n_2}$.

Whenever necessary, we shall indicate the dependence of α_m and α_M on (n_1, n_2, d_1, d_2) by writing $\alpha_m = \alpha_m(n_1, n_2, d_1, d_2)$, and similarly for α_M.

Within the allowed range for α, there is a discrete set of *critical values*. These are the values of α for which it is numerically possible to have a subtriple $T' = (E_1', E_2', \phi')$ such that $\mu(E_1' \oplus E_2') \neq \mu(E_1 \oplus E_2)$ but $\mu_\alpha(T') = \mu_\alpha(T')$. All other values of α are called *generic*. The critical values of α are precisely the values for α at which the stability properties of a triple can change, i.e. there can be triples which are strictly α-semistable, but either α'-stable or α'-unstable for $\alpha' \neq \alpha$.

The following result relates the stability conditions for holomorphic triples and that for $\mathrm{U}(p,q)$-Higgs bundles.

Proposition 4.8. *A $\mathrm{U}(p,q)$-Higgs bundle (E, Φ) with $\beta = 0$ or $\gamma = 0$ is (semi)stable if and only if the corresponding holomorphic triple is α-(semi)stable for $\alpha = 2g - 2$.*

Combining Propositions 3.15 and 4.8, we have the following characterization of the subspace of local minima $\mathcal{N}(p, q, a, b)$.

Theorem 4.9. *Let $\mathcal{N}(p, q, a, b)$ be the subspace of local minima of f on $\mathcal{M}(p, q, a, b)$, and let τ be the Toledo invariant.*

If $a/p < b/q$, or equivalently if $\tau < 0$, then $\mathcal{N}(p, q, a, b)$ can be identified with the moduli space of α-polystable triples of type $(p, q, a + p(2g - 2), b)$, with $\alpha = 2g - 2$.

If $a/p > b/q$, or equivalently if $\tau > 0$, then $\mathcal{N}(p,q,a,b)$ can be identified with the moduli space of α-polystable triples of type $(q, p, b+q(2g-2), a)$, with $\alpha = 2g-2$.

That is,

$$\mathcal{N}(p,q,a,b) \cong \begin{cases} \mathcal{N}_{2g-2}(p,q,a+p(2g-2),b) & \text{if } a/p < b/q \text{ (equivalently } \tau < 0) \\ \mathcal{N}_{2g-2}(q,p,b+q(2g-2),a) & \text{if } a/p > b/q \text{ (equivalently } \tau > 0) \end{cases}$$

In view of Theorem 4.9, it is important to understand where $2g-2$ lies in relation to the range (given by Proposition 4.7) for the stability parameter α. One has the following.

Proposition 4.10. *Fix (p,q,a,b). Then*

$$0 \leqslant |\tau| \leqslant \min\{p,q\}(2g-2) \Leftrightarrow 0 < \alpha_m(p,q,a,b) \leqslant 2g-2 \leqslant \alpha_M(p,q,a,b) \quad \text{if } p \neq q \tag{4.6}$$

Proposition 4.10 shows that in order to study $\mathcal{N}(p,q,a,b)$ for different values of the Toledo invariant, we need to understand the moduli spaces of triples for values of α that may lie anywhere (including at the extremes α_m and α_M) in the α-range given in Proposition 4.7.

We can assume $n_1 > n_2$, since by triples duality one has the following.

Proposition 4.11. $\mathcal{N}_\alpha(n_1,n_2,d_1,d_2) \cong \mathcal{N}_\alpha(n_2,n_1,-d_2,-d_1)$.

Recall that the allowed range for the stability parameter is $\alpha_m \leqslant \alpha \leqslant \alpha_M$, where $\alpha_m = \mu_1 - \mu_2$ and $\alpha_M = \frac{2n_1}{n_1-n_2}\alpha_m$, and we assume that $\mu_1 - \mu_2 > 0$. We describe the moduli space $\mathcal{N}_\alpha$ for $2g-2 \leqslant \alpha < \alpha_M$.

Let α_L be the largest critical value in (α_m, α_M), and let $\mathcal{N}_L$ (respectively $\mathcal{N}_L^s$) denote the moduli space of α-polystable (respectively α-stable) triples for $\alpha_L < \alpha < \alpha_M$. We refer to $\mathcal{N}_L$ as the 'large α' moduli space.

Proposition 4.12. *Let $T = (E_1, E_2, \phi)$ be an α-semistable triple for some α in the range $\alpha_L < \alpha < \alpha_M$. Then T is of the form*

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0,$$

with F locally free, and E_2 and F are semistable.

In the converse direction, we have[5]:

Proposition 4.13. *Let $T = (E_1, E_2, \phi)$ be a triple of the form*

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow F \longrightarrow 0, \tag{4.7}$$

with F locally free and such that the extension is non-trivial. If E_2 and F are stable then T is α-stable for α in the range $\alpha_L < \alpha < \alpha_M$.

[5] This proposition replaces Proposition 7.6 of [6]. We thank Stefano Pasotti, Francesco Prantil and Carlos Tejero for pointing out errors in this proposition.

Proof. Regarding the top line of the diagram

$$
\begin{array}{ccc}
E_2 & \longrightarrow & \phi(E_2) \\
\downarrow & & \downarrow \\
E_2 & \xrightarrow{\phi} & E_1 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & F
\end{array}
\tag{4.8}
$$

as a subtriple T' of T, and the bottom line as a quotient triple T'', we can consider T as an extension of triples

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0.$$

It follows from the stability of E_2 that the subtriple T' is α-stable for any $\alpha > 0$. It follows similarly from the stability of F, that the quotient triple T'' is α-stable for any α. In particular, T' and T'' are both α_M-stable.

A simple calculation shows that

$$\mu_{\alpha_M}(T') = \mu(E_2) + \frac{1}{2}\alpha_M = \mu(F) = \mu_{\alpha_M}(T''). \tag{4.9}$$

It is a general fact that an extension of α-semistable triples of the same α-slope is itself α-semistable. Thus, we deduce from (4.9) that the triple T is α_M-semistable.

It remains to show that T is α-stable for $\alpha_L < \alpha < \alpha_M$. We do this by showing that there is no α-destabilizing subtriple, i.e., a subtriple S of T such that $\mu_\alpha(S) \geqslant \mu_\alpha(T)$.

To do this, we first observe that the extension (4.8) is a Jordan–Hölder filtration of T considered as an α_M-semistable object. This follows since T' and T'' are α_M-stable and have the same α_M-slope. Hence the associated graded object for T in the category of α_M-semistable triples is

$$\mathrm{Gr}(T) = T' \oplus T''. \tag{4.10}$$

Now assume that $S \subseteq T$ is α-destabilizing for α in the range $\alpha_L < \alpha < \alpha_M$. By the continuity of $\mu_\alpha(S)$ in α, we have $\mu_{\alpha_M}(S) \geqslant \mu_{\alpha_M}(T)$ and hence α_M-semistability of T implies that $\mu_{\alpha_M}(S) = \mu_{\alpha_M}(T)$. It follows that in $\mathrm{Gr}(T) = T' \oplus T''$, the triple S must be isomorphic to either T' or T''.

We first show that S cannot be isomorphic to T', i.e, that the subtriple T' is not destabilizing for $\alpha < \alpha_M$. The key piece of information is that

$$\frac{n_2(T')}{n(T')} = \frac{n_2}{2n_2} = \frac{1}{2},$$
$$\frac{n_2(T)}{n(T)} = \frac{n_2}{n_1 + n_2} < \frac{1}{2},$$

where, for any triple $T = (E_2, E_1, \phi)$, we write $n_i(T) = \mathrm{rk}(E_i)$ and $n(T) = \mathrm{rk}(E_2 \oplus E_1)$. Hence

$$\frac{n_2(T')}{n(T')} > \frac{n_2(T)}{n(T)} > \frac{n_2(T'')}{n(T'')}. \tag{4.11}$$

But, since $\mu_{\alpha_M}(T') = \mu_{\alpha_M}(T)$,

$$\mu_\alpha(T') - \mu_\alpha(T) = (\alpha - \alpha_M)\left(\frac{n_2(T')}{n(T')} - \frac{n_2}{n}\right) < 0$$

for $\alpha < \alpha_M$.

Finally we show that S cannot be isomorphic to T''. In fact, if T has a subtriple isomorphic to T'', then E_1 has a subbundle, $\tilde{F}$, isomorphic to F. The composition of the isomorphism from F to $\tilde{F}$ with the projection from E_1 to F produces a homomorphism

$$\psi : F \to F.$$

Since F is stable, ψ is either zero or a multiple of the identity. If it is zero, then there must be a non-trivial homomorphism from F to E_2. This is impossible since $\mu(\tilde{F}) > \mu(E_2)$, and both are stable bundles. Hence the the isomorphism from F to $\tilde{F}$ splits the extension (4.7). But this contradicts our assumptions. □

Theorem 4.14. *Assume that $n_1 > n_2$ and $d_1/n_1 > d_2/n_2$.*
Then the moduli space $\mathcal{N}_L^s = \mathcal{N}_L^s(n_1, n_2, d_1, d_2)$ is smooth of dimension

$$(g-1)(n_1^2 + n_2^2 - n_1 n_2) - n_1 d_2 + n_2 d_1 + 1,$$

and includes a $\mathbb{P}^N$-fibration $\mathcal{P}$ over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$, where $M^s(n, d)$ is the moduli space of stable bundles of rank n and degree d, and $N = n_2 d_1 - n_1 d_2 + n_1(n_1 - n_2)(g-1) - 1$. Moreover, the complex codimension of $\mathcal{N}_L^s \setminus \mathcal{P}$ is at least $g - 1$. In particular, $\mathcal{N}_L^s(n_1, n_2, d_1, d_2)$ is non-empty and irreducible.

If $\mathrm{GCD}(n_1 - n_2, d_1 - d_2) = 1$ *and* $\mathrm{GCD}(n_2, d_2) = 1$*, then $\mathcal{N}_L^s(n_1, n_2, d_1, d_2)$ is isomorphic to $\mathcal{P}$.*

Proof. The birational equivalence between $\mathcal{P}$ and $\mathcal{N}_L^s$ is proved in [6]. To obtain the precise estimate of the codimension of $\mathcal{N}_L^s \setminus \mathcal{P}$ in $\mathcal{N}_L^s$ we see that, by Proposition 4.12, it suffices to estimate the dimension of stable triples like (4.7) with E_2 and F semistable.

Now, for any family of semistable bundles, the complex codimension of the set of strictly semistable bundles is at least $g - 1$. A computation of the precise estimate can be found in [8]. The proof is finished by observing that for a stable triple of the form (4.7), $H^0(X, E_2 \otimes F^*) = 0$ (see [6]). □

The following is proved in [6].

Theorem 4.15. *Let α be any value in the range $\alpha_m < 2g - 2 \leqslant \alpha < \alpha_M$. Then $\mathcal{N}_\alpha^s$ is birationally equivalent to $\mathcal{N}_L^s$. Moreover, they are isomorphic outside of a set of complex codimension greater or equal than $g - 1$. In particular, $\mathcal{N}_\alpha^s$ is non-empty and irreducible.*

4.3. Homotopy groups of moduli spaces of triples

The strategy to compute the homotopy groups of $\mathcal{N}(p, q, a, b)$ is to compute first those of the moduli space of α-stable triples $\mathcal{N}_\alpha^s$ for large α.

Let $n_1 > n_2$, and let $\mathcal{P} \subset \mathcal{N}_L$ be the $\mathbb{P}^N$-fibration over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$ given in Theorem 4.14. As a consequence of Theorems 4.14 and 4.15, we have the following.

Proposition 4.16. *Let $2g-2 \leqslant \alpha < \alpha_M$. Then*

$$\pi_i(\mathcal{N}_\alpha^s(n_1,n_2,d_1,d_2)) \cong \pi_i(\mathcal{N}_L^s(n_1,n_2,d_1,d_2)) \cong \pi_i(\mathcal{P}) \quad \text{for } i \leqslant 2g-4.$$

Associated with the $\mathbb{P}^N$-fibration $\mathcal{P}$ over $M^s(n_1-n_2,d_1-d_2) \times M^s(n_2,d_2)$, there is a homotopy sequence

$$\begin{aligned}\cdots \longrightarrow \pi_i(\mathbb{P}^N) \longrightarrow \pi_i(\mathcal{P}) \longrightarrow \pi_i(M^s(n_2,d_2))&\\ \times \pi_i(M^s(n_1-n_2,d_1-d_2)) \longrightarrow \pi_{i-1}(\mathbb{P}^N) \longrightarrow \cdots&\end{aligned} \tag{4.12}$$

Proposition 4.17. *Let $n_1 > n_2$ and $n_2 d_1 > n_1 d_2$. Assume that $(n_2,g) \neq (2,2)$ and $(n_1-n_2,g) \neq (2,2)$ (for our applications, we will actually assume $g \neq 3$). Then*

(1) $\pi_1(\mathcal{P}) \cong \pi_1(M^s(n_2,d_2)) \times \pi_1(M^s(n_1-n_2,d_1-d_2)) \cong H_1(X,\mathbb{Z}) \oplus H_1(X,\mathbb{Z})$;
(2) $\pi_2(\mathcal{P})$ is the middle term in an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(\mathcal{P}) \longrightarrow Q(n_1,n_2,d_1,d_2) \longrightarrow 0$$

where

$$Q(n_1,n_2,d_1,d_2) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_{\mathrm{GCD}(n_2,d_2)} & \\ \quad \oplus\,\mathbb{Z}_{\mathrm{GCD}(n_1-n_2,d_1-d_2)} & \text{if } n_2>1 \text{ and } (n_1-n_2)>1\\ \mathbb{Z}\oplus\mathbb{Z}_{\mathrm{GCD}(n_2,d_2)} & \text{if } n_2>1 \text{ and } (n_1-n_2)=1\\ \mathbb{Z}\oplus\mathbb{Z}_{\mathrm{GCD}(n_1-n_2,d_1-d_2)} & \text{if } n_2=1 \text{ and } (n_1-n_2)>1\\ 0 & \text{if } n_2=1 \text{ and } n_1=2\end{cases}$$

Remark 4.18. It follows immediately from the exact sequence in (2) of Proposition 4.17 that the free part of the finitely generated Abelian group $\pi_2(\mathcal{P})$ equals the direct sum of $\mathbb{Z}$ and the free part of $Q(n_1,n_2,d_1,d_2)$. In particular, if the co-primality conditions $\mathrm{GCD}(n_2,d_2)=1$ and $\mathrm{GCD}(n_1-n_2, n_2-d_2)=1$ hold, then we have a complete description of $\pi_2(\mathcal{P})$ as the direct sum $\mathbb{Z}\oplus Q(n_1,n_2,d_1,d_2)$. Also, under any circumstances, it follows that $\pi_2(\mathcal{P})\otimes\mathbb{Q} \cong \mathbb{Q}\oplus Q(n_1,n_2,d_1,d_2)\otimes\mathbb{Q}$. So, for a rational homotopy, our results are complete.

Proof of Proposition 4.17. From the homotopy sequence (4.12), since $\pi_0(\mathbb{P}^N) = \pi_1(\mathbb{P}^N) = 0$, we deduce that $\pi_1(\mathcal{P}) \cong \pi_1(M^s(n_2,d_2)) \times \pi_1(M^s(n_1-n_2,d_1-d_2))$. Statement (1) follows from Theorem 4.2.

Since $\pi_1(\mathbb{P}^N)=0$, (4.12) gives

$$\cdots \longrightarrow \pi_2(\mathbb{P}^N) \longrightarrow \pi_2(\mathcal{P}) \longrightarrow \pi_2(M^s(n_2,d_2)) \times \pi_2(M^s(n_1-n_2,d_1-d_2)) \longrightarrow 0. \tag{4.13}$$

On the other hand, by Hurewicz' theorem, $\pi_2(\mathbb{P}^N) \cong H_2(\mathbb{P}^N,\mathbb{Z}) \cong \mathbb{Z}$. Now, the map $f : \mathbb{Z} \cong \pi_2(\mathbb{P}^N) \longrightarrow \pi_2(\mathcal{P})$ in (4.13) is injective, since one has the commutative diagram

$$\begin{tikzcd} \mathbb{Z}\cong\pi_2(\mathbb{P}^N) \arrow[r, "f"] \arrow[d, "\Vert"'] & \pi_2(\mathcal{P}) \arrow[d] \\ H_2(\mathbb{P}^N,\mathbb{Z}) \arrow[r] & H_2(\mathcal{P},\mathbb{Z}). \end{tikzcd}$$

Also $H_2(\mathbb{P}^N, \mathbb{Z}) \longrightarrow H_2(\mathcal{P}, \mathbb{Z})$ must be injective, because the restriction of an ample line bundle over $\mathcal{P} \subset \mathcal{N}_L$ to $\mathbb{P}^N$ must give an ample line bundle. Note that the natural map $H_2(\mathbb{P}^N) \longrightarrow H_2(\mathcal{N}_L)$ is injective and factors through $H_2(\mathbb{P}^N) \longrightarrow H_2(\mathcal{P}) \longrightarrow H_2(\mathcal{N}_L)$. Now, we obtain (2) from Theorem 4.2 and the fact that if $n = 1$ then the moduli space $M^s(n, d)$ is the Jacobian of degree d line bundles. □

As a corollary of Propositions 4.16 and 4.17, we have the following.

Theorem 4.19. *Assume $n_1 > n_2$, $n_2 d_1 > n_1 d_2$, $g \geqslant 3$, and $2g - 2 \leqslant \alpha < \alpha_M$. Then*

(1) $\pi_1(\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)) \cong H_1(X, \mathbb{Z}) \oplus H_1(X, \mathbb{Z})$;
(2) $\pi_2(\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2))$ *is the middle term in an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)) \longrightarrow Q(n_1, n_2, d_1, d_2) \longrightarrow 0$$

where

$$Q(n_1, n_2, d_1, d_2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{\mathrm{GCD}(n_2, d_2)} & \\ \quad \oplus \mathbb{Z}_{\mathrm{GCD}(n_1 - n_2, d_1 - d_2)} & \text{if } n_2 > 1 \text{ and } (n_1 - n_2) > 1 \\ \mathbb{Z} \oplus \mathbb{Z}_{\mathrm{GCD}(n_2, d_2)} & \text{if } n_2 > 1 \text{ and } (n_1 - n_2) = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_{\mathrm{GCD}(n_1 - n_2, d_1 - d_2)} & \text{if } n_2 = 1 \text{ and } (n_1 - n_2) > 1 \\ 0 & \text{if } n_2 = 1 \text{ and } n_1 = 2 \end{cases}$$

Remark 4.20. Theorem 4.19 gives a complete description of $\pi_2(\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2))$ when the co-primality conditions $\mathrm{GCD}(n_2, d_2) = 1$ and $\mathrm{GCD}(n_1 - n_2, n_2 - d_2) = 1$ hold, and of $\pi_2(\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)) \otimes \mathbb{Q}$ under all circumstances (cf. Remark 4.18).

Using the results of [3], a complete description of $\pi_2(\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2))$ can also be given in the case when $n_2 = 1$ and $n_1 - n_2 > 1$, as we now explain. In that paper, the moduli space of *stable pairs* (V, ϕ) was studied. Here, V is a vector bundle and $\phi \in H^0(X, V)$ is a holomorphic section of V. Viewing the section ϕ as a map $\phi\colon \mathcal{O} \to V$, a pair (V, ϕ) gives rise to a triple $(E_1, E_2, \phi) = (V, \mathcal{O}, \phi)$. Through this correspondence, it can be seen that the moduli space of triples $\mathcal{N}_\alpha^s$ of triples with $n_2 = 1$ fibres over the Jacobian variety of the curve, with fibres isomorphic to the moduli space of pairs. Among other things, in [3] the second homotopy group of the moduli space of pairs was calculated to be $\mathbb{Z} \oplus \mathbb{Z}$ for α between α_m and the first critical value of α larger than α_m. It follows from these results that, when $d_2 = 1$, one has $\pi_2(\mathcal{N}_\alpha^s) = \mathbb{Z} \oplus \mathbb{Z}$ for such α. Combining this fact with Proposition 4.16, it follows that $\pi_2(\mathcal{N}_\alpha^s) = \mathbb{Z} \oplus \mathbb{Z}$ for $2g - 2 \leqslant \alpha < \alpha_M$ in the case $n_2 = 1$ and $n_1 - n_2 > 1$.

4.4. Homotopy groups of $\mathcal{M}(p, q, a, b)$

Combining Propositions 3.2, 3.12 and 3.15, we have the following.

Theorem 4.21. *Let* $\mathrm{GCD}(p + q, a + b) = 1$. *Then*

$$\pi_i(\mathcal{M}(p, q, a, b)) \cong \pi_i(\mathcal{N}(p, q, a, b)), \quad \text{for } i \leqslant 2g - 4.$$

As a corollary of Theorems 4.21, 4.9 and 4.19 and Proposition 4.11, we conclude the following.

Theorem 4.22. *Let $p \neq q$ and* $\mathrm{GCD}(p + q, a + b) = 1$, *and let $g \geqslant 3$. Then*

(1) $\pi_1(\mathcal{M}(p, q, a, b)) \cong H_1(X, \mathbb{Z}) \oplus H_1(X, \mathbb{Z})$;

(2) $\pi_2(\mathcal{M}(p,q,a,b))$ *is the middle term in an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(\mathcal{M}(p,q,a,b)) \longrightarrow Q(n_1,n_2,d_1,d_2) \longrightarrow 0,$$

where

$$(n_1,n_2,d_1,d_2) = \begin{cases} (p,q,a+p(2g-2),b) & \text{if } \tau<0 \text{ and } p>q \\ (q,p,-b,-a-p(2g-2)) & \text{if } \tau<0 \text{ and } p<q \\ (p,q,-a,-b-q(2g-2)) & \text{if } \tau>0 \text{ and } p>q \\ (q,p,b+q(2g-2),a) & \text{if } \tau>0 \text{ and } p<q \end{cases}$$

and where $Q(n_1,n_2,d_1,d_2)$ *is as in Theorem* 4.19.

Remark 4.23. Theorem 4.22 gives a complete description of $\pi_2(\mathcal{M}(p,q,a,b))$ when the co-primality conditions $\mathrm{GCD}(n_2,d_2)=1$ and $\mathrm{GCD}(n_1-n_2,n_2-d_2)=1$ hold, and of $\pi_2(\mathcal{M}(p,q,a,b))\otimes\mathbb{Q}$ under all circumstances (cf. Remarks 4.18 and 4.20).

Remark 4.24. As a consequence of Theorem 4.21 and the connectedness of $\mathcal{N}(p,q,a,b)$, we have that $\mathcal{M}(p,q,a,b)$ is also connected, as proved in [5].

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